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Sample path properties of Poissonian type stochastic integral processes are studied. It is proven that various properties of the sections of the deterministic kernel (as, for example, unboundedness, discontinuity, etc.) are inherited by the sample paths of the corresponding stochastic integral process. An analogous statement for Gaussian processes is false. As a main tool, a series representation of stochastic integral processes is fully developed and this may be of independent interest.

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INFINITELY DIVISIBLE PROCESSES[1]

by

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Abstract: Let $\{X(t): t\epsilon T\}$ be a stochastic process equal in distribution to $\{\int_S f(t,s)\Lambda(ds): t\epsilon T\}$, where Λ is a symmetric independently scattered random measure and f is a suitable deterministic function. It is shown that various properties of the sections $f(\cdot,s)$, $s \epsilon S$, are inherited by the sample paths of X, provided X has no Gaussian component. The analogous statement for Gaussian processes is false. As a main tool, a "LePage-type" series representation is fully developed for symmetric stochastic integral processes and this may be of independent interest.

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1. Introduction.

A stochastic process $X = \{X(t): t\epsilon T\}$, with an arbitrary index set T, is said to be *infinitely divisible* (i.d.) if its finite dimensional distributions are all i.d. An i.d. process X is said to be a *stochastic integral process* if

$$(*) \qquad \{X(t): t\epsilon T\} \overset{d}{=} \{\textstyle\int_S f(t,s)\ \Lambda(ds): t \epsilon T\},$$

where $f:T{\times}S \to \mathbb{R}(\mathbb{C})$ is a deterministic function and $\Lambda = \{\Lambda(A): A \epsilon \mathcal{S}\}$ is an independently scattered i.d. random measure on a δ-ring $\mathcal{S}$ of subsets of a certain set S. Here " $\overset{d}{=}$ " denotes equality in (all finite-dimensional) distribution(s). The equality $(*)$ will be referred to as a stochastic integral representation of X. The family of stochastic integral processes contains such important i.d. processes as harmonizable, moving averages, fractional processes, strictly stable and semistable, and also the so-called ξ-radial processes, recently introduced and studied by M.B. Marcus [2].

In this paper we establish a connection between certain sample path properties of stochastic integral processes (satisfying $(*)$) and the corresponding properties of section of $f(\cdot,s)$, $s \epsilon S$. In Theorem 4 (Section 4) we show that the lack of certain analytic regularities of the sections of $f(\cdot,s)$, $s \epsilon S$, (as for example: discontinuity, unboundedness, etc.) is inherited by the sample paths of symmetric stochastic integral processes without Gaussian component. The analogous statement for Gaussian processes is false as is illustrated by an example. Integrability of sample paths is studied in Theorem 6 (Section 4), where we use a kind of Monte-Carlo technique to show that the sections $f(\cdot,s)$, $s \epsilon S$, must have at least the same order of integrability as the paths of X.

One way of looking at these results is that they provide *immediately verifiable* necessary conditions for interesting sample path properties. Therefore one may easily *exclude* a number of path properties that do not hold

and this gives an insight into the behaviour of the sample paths of the process.

The main tool used in this paper is the series representation of stochastic integral processes obtained in Section 3. This series representation, which we derive as a special case of a "generalized shot noise" (see [7]), generalizes various "LePage type" representations of symmetric stochastic integral processes (see Examples in Section 3).

In Section 2 we give the pertinent facts concerning random measures and stochastic integrals. Further details can be found in the work of B.S. Rajput and the author [5]. In Section 5 we discuss possible generalizations of the results of Section 4 to not necessarily symmetric stochastic integral processes.

Notation. We shall introduce now some notations that will be used throughout this paper. A stochastic process $\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}$ is said to be an *independently scattered i.d. random measure* (i.d. random measure, for short) if

(a) for every pairwise disjoint $A_1, A_2, \ldots \in \mathcal{S}$, the random variables $\Lambda(A_1), \Lambda(A_2), \ldots$ are independent and

$$\Lambda(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \Lambda(A_n) \quad \text{a.s.,}$$

provided $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$;

(b) for every $A \in \mathcal{S}$, $\Lambda(A)$ has an i.d. distribution.

Further, an i.d. random measure Λ is said to be *symmetric* if $-\Lambda(A) \overset{d}{=} \Lambda(A)$, for every $A \in \mathcal{S}$. A typical and important example of an i.d. random measure is the random measure generated by the increments of a Lévy process, say $\{Z(s) : s \in S\}$, where S is a (possibly unbounded) interval. By definition $\Lambda((a,b])=Z(b)-Z(a)$, $(a,b] \subset S$, and $\mathcal{S}$ is the family of bounded Borel subsets of S.

From now on we shall assume that the following condition is satisfied: there exists a sequence $\{S_n\}_{n=1}^{\infty} \subset \mathcal{S}$ such that $\bigcup_{n=1}^{\infty} S_n = S$. A set $A \in \sigma(\mathcal{S})$ is said to be a Λ-zero set if $\Lambda(A_1) = 0$ a.s. for every $A_1 \subset A$, $A_1 \in \mathcal{S}$. A σ-finite

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measure λ on $\sigma(\mathcal{G})$ is said to be a *control measure of* Λ if Λ and λ have the same families of zero sets. In the case of the random measure generated by the increments of a Lévy process, λ may be chosen as the Lebesgue measure, but every other measure equivalent to the Lebesgue measure is also a control measure of Λ. An explicit form of a control measure for a general Λ is given in Proposition 2.1(c) [5].

To avoid obvious difficulties with the measurability of certain sets (see Theorem 4) it is convenient to assume the separability of the representation (∗). The definition given below parallels Doob's definition of separability of stochastic processes. Let T be a separable metric space. The representation (∗) is said to be *separable* if there exists a sequence $\{t_n\}_{n=1}^{\infty} \subset T$ and a Λ-zero set $S_0 \subset S$ with the property: for every $t \in T$ there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\lim_{k \to \infty} f(t_{n_k}, s) = f(t, s)$, for every $s \in S \setminus S_0$. For example, if the sections $f(\cdot, s)$, $s \in S$, are continuous (or one-sided continuous if $T \subset \mathbb{R}$), then the representation (∗) is separable. As in the case of stochastic processes, the separability of the stochastic integral representation is a minimal assumption which can always be made, without loss of generality. Indeed, let $\lambda^{(1)}$ be a probability measure on $\sigma(\mathcal{G})$, equivalent to λ. Then $f=\{f(t, \cdot): t \in T\}$ may be viewed as a stochastic process and by the Doob's theorem there exists a separable modification of f with values in a compactification of $\mathbb{R}$ ($\mathbb{C}$). Such a modification does not affect (∗), which completes the argument.

In this paper, $(\Omega, \mathcal{F}, P)$ will denote a probability space, $\mathcal{L}(Z)$ the distribution of a random element Z and Leb the Lebesgue measure on $\mathbb{R}$. For simplicity we will consider only real f's in (∗), but the results extend easily to the complex case (see the Remark concluding Section 3).

2. Symmetric i.d. random measures and stochatic integrals

In this section we give the pertinent formulas and facts concerning stochastic integrals relative to i.d. random measures. These formulas will be used in Section 3 to derive series representations of symmetric stochastic integral processes

Let $\Lambda = \{\Lambda(A): A \in \mathcal{S}\}$ be a symmetric i.d. random measure without Gaussian component. The characteristic function of $\Lambda(A)$ can be written in Lévy's form:

$$(2.1) \qquad \hat{\mathcal{L}}(\Lambda(A))(u) = \exp\{2 \int_0^\infty (\cos ux-1)F_A(dx)\},$$

$u \in \mathbb{R}$, where F_A is a (symmetric) Lévy measure on $\mathbb{R}$. Let λ be an arbitrary but fixed control measure of Λ. We shall show that there exists a measurable family $\{q(s,\cdot)\}_{s \in S}$ of Borel measures on $(0,\infty)$ such that

$$(2.2) \qquad 2F_A(B) = \int_A [\int_0^\infty I_B(x)q(s,dx)]\lambda(ds),$$

for every Borel set $B \subset (0,\infty)$ and $A \in \mathcal{S}$; furthermore

$$(2.3) \qquad \lambda(\{s \in S: q(s,(0,\infty)) = 0\}) = 0,$$

and for every $s \in S$,

$$(2.4) \qquad \int_0^\infty (1 \wedge x^2)q(s,dx) < \infty.$$

Indeed, as it was shown in Propositions 2.1 and Lemma 2.3 of [5], the measure v determined by $v(A) = \int_{\mathbb{R}}(1 \wedge x^2)F_A(dx)$, $A \in \mathcal{S}$, is a control measure of Λ and, furthermore, there exists a measurable family $\{\rho(s,\cdot)\}_{s \in S}$ of Lévy measures on $\mathbb{R}$ such that

$$(2.5) \qquad F_A(B) = \int_A [\int_{\mathbb{R}} I_B(x)\rho(s,dx)]v(ds),$$

$A \in \mathcal{S}$, $B \subset \mathbb{R}$. Since λ and v are equivalent σ-finite measures on $\sigma(\mathcal{S})$, there exists a strictly positive and finite everywhere version ψ of the Radon-Nikodym derivative $dv/d\lambda$. Put $q(s,dx) = 2\psi(s)\rho(s,dx)$, $s \in \mathcal{S}$, $x > 0$. Then (2.2) follows from (2.5). Since $\rho(s,\cdot)$ is a Lévy measure, (2.4) is satisfied. Finally note that $A_0 = \{s:q(s,(0,\infty)) = 0\}$ is a Λ-zero set by (2.2) and (2.1), so that $\lambda(A_0) = 0$. We have proven (2.2)-(2.4).

Using (2.2), (2.1) can be rewritten in the form:

$$\hat{\mathscr{L}}(\Lambda(A))(u) = \exp\{\int_A [\int_0^\infty [\cos ux - 1]q(s,dx)]\lambda(ds)\}.$$

Set

$$\Phi(u,s) = \int_0^\infty [1^\wedge(ux)^2]q(s,dx),$$

$s \in S$, $u \in \mathbb{R}$, and

$$L_\Phi = \{g: S \to \mathbb{R}: g \text{ is } \sigma(\mathscr{S})\text{-measurable and } \int_S \Phi(g(s),s)\lambda(ds) < \infty\}.$$

Then L_Φ is a linear metric space (a so-called Musielak–Orlicz space) with the F-norm defined by

$$||g||_\Phi = \inf\{c > 0: \int_S \Phi(c^{-1}g(s),s)\lambda(ds) \leq c\}.$$

The next proposition, which follows as a particular case from Theorem 3.4 and Proposition 3.6(i) in [5], states the basic facts regarding Wiener-type stochastic integrals relative to symmetric i.d. random measures.

PROPOSITION 1. *There exists a unique isomorphism, denoted by* $\int_S (\cdot)d\Lambda$, *from* L_Φ *into* $L_0(\Omega,\mathscr{F},P)$ *such that*

$$\int_S (\sum_{j=1}^n a_j I_{A_j})d\Lambda = \sum_{j=1}^n a_j \Lambda(A_j) \quad \text{a.s.}$$

for every $n \geq 1$, $a_1,\ldots,a_n \in \mathbb{R}$ *and pairwise disjoint* $A_1,\ldots,A_n \in \mathscr{S}$. *Further, the characteristic function of* $\int_S g d\Lambda$ *is given by*

$$(2.6) \qquad \hat{\mathscr{L}}(\int_S g d\Lambda)(u) = \exp\{\int_S [\int_0^\infty (\cos(uxg(s))-1)q(s,dx)]\lambda(ds)\}.$$

3. Series developments of symmetric stochastic integral processes.

Let Λ,λ,q be as in the previous section, so that (2.1)–(2.4) hold. Set

$$R(u,s) = \inf\{x > 0: q(s,(x,\infty)) \leq u\}, \quad u > 0,$$

(in words: for fixed s, $R(\cdot,s)$ is the right continuous inverse of the function $x \to q(s,(x,\infty)))$. Let $\lambda^{(1)}$ be an arbitrary probability measure on $(S,\sigma(\mathscr{S}))$ equivalent to λ. Put

$$R^{(1)}(u,s) = R(u\frac{d\lambda^{(1)}}{d\lambda}(s),s), \quad u > 0, \ s \in S,$$

where the version of the Radon–Nikodym derivative $d\lambda^{(1)}/d\lambda$ is chosen to be

strictly positive and finite everywhere.

Let $\{\xi_n\}$, $\{e_n\}$, $\{\epsilon_n\}$ be independent sequences of random elements (variables) such that:

-$\{\xi_n\}$ is a sequence of i.i.d. random elements in $(S,\sigma(\mathcal{Y}))$ with $\mathcal{L}(\xi_n) = \lambda^{(1)}$.

-$\{e_n\}$ is a sequence of i.i.d. exponential random variables, i.e. $P\{e_n > x\} = \exp(-x)$, $x > 0$.

-$\{\epsilon_n\}$ is a sequence of i.i.d. random variables with $P\{\epsilon_n=-1\} = P\{\epsilon_n=1\} = \frac{1}{2}$.

Put $\gamma_n = e_1 + \ldots + e_n$.

PROPOSITION 2. *Let $\{X(t): t\epsilon T\}$ be a stochastic process satisfying (*), where Λ is a symmetric i.d. random measure without Gaussian component. Then with the above notations, for every $t\epsilon T$, the series*

$$Y(t) = \sum_{n=1}^{\infty} \epsilon_n R^{(1)}(\gamma_n, \xi_n) f(t, \xi_n)$$

converges a.s. and

$$\{X(t): t\epsilon T\} \overset{d}{=} \{Y(t): t\epsilon T\}.$$

Proof. Let $g \epsilon L_\phi$ (recall Proposition 1). First we shall show that

$$(3.1) \qquad \int_S g\, d\Lambda \overset{d}{=} \sum_{n=1}^{\infty} \epsilon_n R^{(1)}(\gamma_n, \xi_n) g(\xi_n),$$

where the series converges a.s. Indeed, this series can be written as a particular case of a generalized shot noise (see [7]):

$$(3.2) \qquad \sum_{n=1}^{\infty} H(\gamma_n, \tilde{\xi}_n),$$

where $\tilde{\xi}_n = (\epsilon_n, \xi_n)$ are i.i.d. random elements in $\tilde{S} = \{-1,1\}\times S$ and $H(u,v) = \epsilon R^{(1)}(u,s)g(s)$, $u>0$, $v=(\epsilon,s) \epsilon \tilde{S}$. In order to establish the convergence and distribution in (3.2) we shall verify the conditions of Theorem 2.4 in [7]. First we need to show that

$$\tilde{G}(B) = \int_{\tilde{S}} \int_0^{\infty} I_{B\backslash\{0\}}(H(u,v))du\tilde{\lambda}(dv), \quad B \subset \mathbb{R},$$

is a Lévy measure, where

$$\widetilde{\lambda} = \mathcal{L}(\widetilde{\xi}_n) = (\tfrac{1}{2}\delta_{-1} + \tfrac{1}{2}\delta_1) \times \lambda^{(1)}.$$

Observe that

(3.3)
$$\widetilde{G}(B) = \tfrac{1}{2}G(-B) + \tfrac{1}{2}G(B),$$

where

$$G(U) = \int_S [\int_0^\infty I_{U\setminus\{0\}}(R^{(1)}(u,s)g(s))du]\lambda^{(1)}(ds).$$

Since, for every $x \geq 0$ and $s \in S$,

(3.4)
$$\text{Leb}\,(\{u{>}0\colon R^{(1)}(u,s) > x\}) =$$

$$\text{Leb}\,(\{u{>}0\colon R(u\tfrac{d\lambda^{(1)}}{d\lambda}(s),s) > x\}) =$$

$$\text{Leb}\,(\{u{>}0\colon R(u,s) > x\})\,/\,\tfrac{d\lambda^{(1)}}{d\lambda}(s) =$$

$$q(s,(x,\infty))\,\tfrac{d\lambda}{d\lambda^{(1)}}(s),$$

we get

(3.5)
$$G(U) = \int_S [\int_0^\infty I_{U\setminus\{0\}}(xg(s))q(s,dx)]\lambda(ds),$$

for every Borel set $U \subset \mathbb{R}$.

On the other hand, by (2.6), (3.5) and (3.3),

$$\widehat{\mathcal{L}}(\int_S g d\Lambda)(u) = \exp\{\int_\mathbb{R}(\cos(uy) - 1)G(dy)\} =$$

$$\exp\{\int_\mathbb{R}(\cos\,(uy) - 1)\,\widetilde{G}(dy)\},$$

for every $u \in \mathbb{R}$. Hence $\widetilde{G}$ is a Lévy measure and

(3.6)
$$\mathcal{L}(\int_S g d\Lambda) = c_1 \text{Pois}(\widetilde{G}).$$

Since, for every $r > 0$,

$$A(r) = \int_0^r \int_{\widetilde{S}} H(u,v) I_{[-1,1]}(H(u,v))\widetilde{\lambda}(dv)du = 0,$$

(3.1) follows from (3.6) and Theorem 2.4 [7].

Now we shall show that (3.1) implies the conclusion of the theorem.

Indeed, let $t_1,\dots,t_m \in T$ and $a_1,\dots,a_m \in \mathbb{R}$ be arbitrary. Put
$g(s) = \sum_{j=1}^m a_j f(t_j,s)$. Then we get

$$\sum_{j=1}^{m} a_j X(t_j) \stackrel{d}{=} \sum_{j=1}^{m} a_j \int_S f(t_j,s)\Lambda(ds)$$

$$= \int_S g\,d\Lambda$$

$$\stackrel{d}{=} \sum_{n=1}^{\infty} \epsilon_n R^{(1)}(\gamma_n,\xi_n)g(\xi_n)$$

$$= \sum_{j=1}^{m} a_j Y(t_j),$$

which completes the proof.

EXAMPLES.

(i) If $\lambda(S) < \infty$, then $\lambda^{(1)}(A) = \lambda(A)/\lambda(S)$, $A \in \sigma(\mathcal{S})$, may be viewed as a "natural choice" of $\lambda^{(1)}$. In this case $d\lambda^{(1)}/d\lambda \equiv 1/\lambda(S)$, hence $R^{(1)}(u,s) = R(u/\lambda(S),s)$.

(ii) Let $X(t) = \int_{\mathbb{R}} k(t-s)\Lambda(ds)$, $t \in \mathbb{R}$, be a moving average process. In this case Λ is a stationary random measure, so that $\lambda=$Leb on $\mathbb{R}$. Hence $\lambda^{(1)}$ can be any distribution on $\mathbb{R}$ with non-vanishing density φ (e.g. Gaussian, double exponential, etc.). By Proposition 2,

$$Y(t) = \sum_{n=1}^{\infty} \epsilon_n R(\gamma_n\varphi(\xi_n),\xi_n)k(t-\xi_n), \quad t \in \mathbb{R},$$

has the same finite dimensional distributions as $\{X(t): t \in \mathbb{R}\}$.

(iii) If Λ is symmetric α-stable, then $R(u,s) = C\,u^{-1/\alpha}$, where C is a numerical constant. In this case,

$$(3.7) \qquad Y(t) = C \sum_{n=1}^{\infty} \epsilon_n \gamma_n^{-1/\alpha} [\frac{d\lambda^{(1)}}{d\lambda}(\xi_n)]^{-1/\alpha} f(t,\xi_n),$$

$t \in T$ is a version of $\{X(t): t \in T\}$ satisfying (*) (T is an arbitrary set). (3.7) generalizes the representation in Marcus and Pisier [3], which assumes $\lambda(S) < \infty$ and $\lambda^{(1)}$ is chosen as in (i).

If one replaces the sequence $\{\epsilon_n\}$ in (3.7) by a sequence $\{\zeta_n\}$ of i.i.d. zero-mean normal random variables with $E|\zeta_n|^{\alpha} = 1$, then the resulting series will converge a.s., for each t, to, say $Z(t)$. The process $\{Z(t): t \in T\}$ has the same finite dimensional distributions as $\{X(t): t \in T\}$. A proof of this

statement parallels the proof of Proposition 2; the measure $\tilde{\lambda}$, in this case, is defined as the joint distribution of ζ_n and ξ_n, i.e. $\tilde{\lambda} = \mathscr{L}((\zeta_n, \xi_n))$, on $\tilde{S} = \mathbb{R} \times S$; the very special form of $R(u,s)$ is crucial for (3.6) to hold. This generalizes the representation in Marcus and Pisier [3] and gives a conditionally Gaussian representation of symmetric stable processes with non-necessarily finite spectral measures (see Lemma 1.6 [3]).

(iv) Let Λ be a symmetrization of a Poisson point process with intensity measure λ, i.e.

$$\hat{\mathscr{L}}(\Lambda(A))(u) = \exp\{2(\cos u - 1)\lambda(A)\}.$$

Then $q(s,(x,\infty)) = 2$ if $x < 1$ and $= 0$ if $x \geq 1$. Hence $R(u,s) = I_{[0,2]}(u)$, and by Proposition 2,

$$Y(t) = \sum_{n=1}^{\infty} \epsilon_n I(\tau_n \frac{d\lambda^{(1)}}{d\lambda}(\xi_n) \leq 2) f(t, \xi_n),$$

$t \in T$, has the same finite dimensional distributions as $\{X(t): t \in T\}$. This representation is especially interesting when $\lambda(S) = \infty$.

REMARK. Proposition 2 holds true for complex stochastic processes satisfying ($*$) with f *complex* and Λ a *real* symmetric i.d. random measure without Gaussian component. To see this, let $X_1(t) = \mathrm{Re}X(t)$, $X_2(t) = \mathrm{Im}X(t)$, $f_1(t,s) = \mathrm{Re}f(t,s)$, $f_2(t,s) = \mathrm{Im}f(t,s)$. Set $T' = T \times \{1,2\}$ and define, for $t' = (t,k) \in T'$,

$$X'(t') = X_k(t), \quad f'(t',s) = f_k(t,s).$$

Then ($*$) is equivalent to

$$\{X'(t'): t' \in T'\} \overset{d}{=} \{\int_S f'(t',s)\Lambda(ds): t' \in T'\}.$$

By Proposition 2 the stochastic process

$$Y'(t') = \sum_{n=1}^{\infty} \epsilon_n R^{(1)}(\tau_n, \xi_n) f'(t', \xi_n), \quad t' \in T'.$$

is equally distributed with $\{X'(t'): t' \in T'\}$. Hence the complex-valued stochastic process

$$X(t) = X'((t,1)) + iX'((t,2)), \quad t \in T,$$

is equally distributed with

$$Y(t) = Y'((t,1)) + iY'((t,2)), \quad t \in T,$$

as it was claimed.

4. Path properties of symmetric stochastic integral processes

The following proposition is crucial for the proofs of Theorems 4 and 6 in this section.

PROPOSITION 3. *Let $\{X(t): t \in T\}$ be a stochastic process satisfying (*), where Λ is a symmetric i.d. random measure without Gaussian component. Let C be a measurable linear subspace of $\mathbb{R}^\infty$. Suppose that, for some sequence $\{t_n\} \subset T$, with probability one*

$$(X(t_1), X(t_2), \ldots) \in C.$$

Then

$$(f(t_1,\cdot), f(t_2,\cdot), \ldots) \in C \quad \lambda\text{-a.e.},$$

where λ is a control measure of Λ.

Proof. Let $\lambda^{(1)}$ be a probability measure on $(S, \sigma(\mathcal{S}))$ equivalent to λ. Using the series representation from Proposition 2 we get

(4.1) $\quad 0 = 2P\{(X(t_1), X(t_2), \ldots) \notin C\} = 2P\{(Y(t_1), Y(t_2), \ldots) \notin C\} =$

$$= P\{(Y(t_1), Y(t_2), \ldots) \notin C\} + P\{(Y'(t_1), Y'(t_2), \ldots) \notin C\},$$

where $Y'(t) = \sum_{n=1}^{\infty} \epsilon_n' R^{(1)}(\gamma_n, \xi_n) f(t, \xi_n)$, $\epsilon_1' = \epsilon_1$ and $\epsilon_n' = -\epsilon_n$ if $n \geq 2$. Since C is a linear space, the last expression in (4.1) is greater than or equal to

$$P\{\epsilon_1 R^{(1)}(\gamma_1, \xi_1)(f(t_1, \xi_1), f(t_2, \xi_1), \ldots) \notin C\} =$$

$$P\{(f(t_1, \xi_1), f(t_2, \xi_1), \ldots) \notin C, R^{(1)}(\gamma_1, \xi_1) > 0\} =$$

$$\int_S [I_{C^c}((f(t_1, s), f(t_2, s), \ldots)) \int_0^\infty I_{(0,\infty)}(R^{(1)}(u,s)) e^{-u} du] \lambda^{(1)}(ds).$$

This shows that the above integral (over S) is equal to zero and, to complete the proof of the Proposition, it suffices to show that

(4.2) $$\int_0^\infty I_{(0,\infty)}(R^{(1)}(u,s)) e^{-u} du > 0 \quad \lambda^{(1)} - \text{a.e.}$$

Indeed, this integral is equal to zero for a given s if and only if $\text{Leb}(\{u: R^{(1)}(u,s) > 0\}) = 0$, which, in view of (3.4), is equivalent to

$q(s,(0,\infty)) = 0$. Since the last equality may hold only for s belonging to a $\lambda^{(1)}$-zero set (recall (2.3)), (4.2) follows and the proof is complete.

In the statement of the next theorem we shall use the following notation: Let T be a metric space and $g: T \to R$. We shall write

$g \in \mathcal{C}_1$ if g is bounded on T,

$g \in \mathcal{C}_2$ if g is continuous on T,

$g \in \mathcal{C}_3$ if g is uniformly continuous on T,

$g \in \mathcal{C}_4$ if g is Lipschitz continuous on T,

further, when T is a (possibly unbounded) interval

$g \in \mathcal{C}_5$ if g is free of oscillatory discontinuities on T,

$g \in \mathcal{C}_6$ if g is of bounded p^{th} variation on every subinterval of T,

$g \in \mathcal{C}_7$ if g is absolutely continuous on every subinterval of T,

$g \in \mathcal{C}_8$ if g is differentiable on T.

THEOREM 4. *Let T be a σ-compact metric space. Let $\{X(t): t \in T\}$ be a separable stochastic process admitting a separable representation (*), where Λ is a symmetric i.d. random measure without Gaussian component. Let λ be a control measure of Λ and suppose that for some k=1,...,8,*

$$\lambda(\{s \in S: f(\cdot,s) \in \mathcal{C}_k\}) > 0$$

(T is an interval when $k \geq 5$). Then

$$P(\{\omega \in \Omega: X(\cdot,\omega) \in \mathcal{C}_k\}) > 0.$$

Proof. Assume, to the contrary, that

$$(4.3) \qquad\qquad P(\{\omega: X(\cdot,\omega) \in \mathcal{C}_k\}) = 1$$

Let $\{t_n\} \subset T$ be the set in the definition of the separability of representation (*). Proceeding very similarly as in the proof of Theorem 2 in [1], one can find a linear measurable subspace $C_k \subset \mathbb{R}^\infty$ such that

$$\lambda(\{s: (f(t_1,s), f(t_2,s),...) \in C_k\})$$

$$= \lambda(\{s: f(\cdot,s) \in \mathcal{C}_k\})$$

and, by (4.3),

$$P(\{\omega: (X(t_1,\omega), X(t_2,\omega), \ldots) \in C_k\}) = 1.$$

Hence, by Proposition 3,

$$\lambda(\{s: f(\cdot,s) \notin \mathscr{C}_k\}) = 0,$$

which contradicts the assumption of the theorem and ends the proof.

It is rather surprising that Theorem 4 fails in the Gaussian case. To see this, we shall construct a *bounded* a.s., discrete parameter, Gaussian process $\{X(t): t \in T\}$, which satisfies (*) and such that $\{f(t,s): t \in T\}$ is *unbounded*, for every s (note that the separability assumptions are satisfied trivially if T is discrete).

EXAMPLE. Let Λ be the Gaussian measure generated by the increments of a Brownian motion on $S = [0,1]$ (i.e., Λ is a white noise on $[0,1]$). Let $\{h_{n,k}\}$ be the Haar system on $[0,1]$, i.e. $h_{0,0} = 1$ and

$$h_{n,k}(s) = \begin{cases} 2^{n/2} & \text{if } s \in [(k-1)2^n, (2k-1)/2^{n+1}), \\ -2^{n/2} & \text{if } s \in [(2k-1)/2^{n+1}, k/2^n], \\ 0 & \text{otherwise,} \end{cases}$$

$n \geq 1$, $k = 1,\ldots,2^n$. Put $Z_{n,k} = \int_0^1 h_{n,k} d\Lambda$. Since $\{h_{n,k}\}$ is an orthonormal system, $\{Z_{n,k}\}$ are i.i.d. $N(0,1)$ random variables. Set $T = \{(n,k): n \geq 1, k=1,\ldots,2^n\}$ and put, for $t = (n,k)$,

$$f(t,s) = n^{-1}h_{n,k}(s), \qquad 0 \leq s \leq 1,$$
$$X(t) = \int_0^1 f(t,s)\Lambda(ds) = n^{-1}Z_{n,k}.$$

Clearly, $\sup_T |f(t,s)| = \infty$ for every $s \in [0,1]$.

Since

$$\sum_{t \in T} P\{|X(t)| > 2\} = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} P\{|n^{-1}Z_{n,k}| > 2\}$$

$$= \sum_{n=1}^{\infty} 2^n P\{|\tfrac{1}{2}Z_{1,1}| > n\}$$

$$\leq \sum_{n=1}^{\infty} 2^n \exp(-n^2) E\exp(\tfrac{1}{4}Z_{1,1}^2) < \infty.$$

$\sup_{T} |X(t)| < \infty$ a.s. by the Borel–Cantelli Lemma.

REMARKS: (i) When the process $\{X(t): t \in T\}$ obeys the zero-one law (as in the stable or semistable cases), the conclusion of Theorem 4 can be strengthened to:

$$P(\{X(\cdot,\omega) \notin \mathscr{C}_k\}) = 1.$$

(ii) Using Propositions 1 and 3, one can easily generalize Theorem 5.1 of [6] (proven for symmetric stable processes) to arbitrary processes admitting representation (*), with Λ being a symmetric i.d. random measure without Gaussian component.

Now we shall investigate the integrability property. Since, in this case, a natural metric space of sample paths is some L^p-space, which consists of *classes* of equivalent functions, the methods of Theorem 4 can not be used. To resolve this difficulty, we shall use a kind of Monte–Carlo technique based on the following elementary fact (which follows immediately from the Borel–Cantelli Lemma):

LEMMA 5. *Let* Z, Z_1, Z_2, ... *be i.i.d. random variables and* $p \in (0,\infty)$. *Then* $E|Z|^p < \infty$ *if and only if* $n^{-1/p}Z_n \to 0$ *a.s., as* $n \to \infty$.

THEOREM 6. *Let* $(T,\mathscr{A}, \mu)$ *be a* σ-*finite measure space. Let* $\{X(t): t \in T\}$ *be a measurable stochastic process which admits representation* (*) *such that* $f: T \times S \to \mathbb{R}$ *is* $\mathscr{A} \times \sigma(\mathscr{S})$-*measurable and* Λ *is a symmetric random measure without Gaussian component. Let* λ *be a control measure of* Λ *and suppose that, for some* $p \in (0,\infty)$.

$$\int_T |X(t)|^P \mu(dt) < \infty \qquad a.s.$$

Then

$$\int_T |f(t,s)|^P \mu(dt) < \infty$$

for λ-a.a. $s \in S$.

<u>Proof.</u> Assume first that $\mu(T) = 1$. Let $U, U_1, U_2, \ldots$ be i.i.d. random elements in T, which are defined on some auxiliary probability space $(\Omega_1, \mathcal{F}_1, P_1)$, such that $\mathcal{L}(U) = \mu$. Since, for P-a.a. $\omega \in \Omega$,

$$\int_{\Omega_1} |X(U(\omega_1), \omega)|^P P_1(d\omega_1) = \int_T |X(t, \omega)|^P \mu(dt) < \infty,$$

by Lemma 5 we have $n^{-1/P} X(U_n, \omega) \to 0$ P_1-a.s. By Fubini's theorem, for P_1-a.a. $\omega_1 \in \Omega_1$, $n^{-1/P} X(U_n(\omega_1), \cdot) \to 0$ P-a.s., which can be written as follows:

$$(X(t_1), X(t_2), \ldots) \in C \qquad P\text{-a.s.},$$

where $t_n = t_n(\omega_1) = U_n(\omega_1)$ and

$$C = \{(a_n) \in \mathbb{R}^\infty : \lim_{n \to \infty} n^{1/P} a_n = 0\}.$$

Clearly, C is a measurable linear suspace of $\mathbb{R}^\infty$. By Proposition 3

$$(f(t_1, \cdot), f(t_2, \cdot), \ldots) \in C \qquad \lambda\text{-a.e.}$$

Using Fubini's theorem again, we get, for λ-a.a. $s \in S$, $n^{-1/P} f(U_n, s) \to 0$ P_1-a.s., which, by Lemma 1, is equivalent to

$$\int_T |f(t,s)|^P \mu(ds) = \int_{\Omega_1} |f(U(\omega_1), s)|^P P_1(d\omega_1) < \infty.$$

Theorem 2 is proven in the case $\mu(T) = 1$.

In the general case, let $\mu^{(1)}$ be a probability measure equivalent to μ, and let $\psi(t) = (d\mu/d\mu^{(1)})(t)$. Put $f_1(t,s) = \psi^{1/P}(t) f(t,s)$, $X_1(t) = \psi^{1/P}(t) X(t)$. Then $\{X_1(t) : t \in T\}$ is a measurable process such that

$$\{X_1(t) : t \in T\} \overset{d}{=} \{\int_S f_1(t,s) \Lambda(ds) : t \in T\}$$

and

$$\int_T |X_1(t)|^P \mu^{(1)}(dt) = \int_T |X(t)|^P \mu(dt) < \infty \qquad a.s.$$

By the firs part of the proof we have

$$\int_T |f(t,s)|^p \mu(dt) = \int_T |f_1(t,s)|^p \mu^{(1)}(dt) < \infty.$$

for λ-a.a. $s \epsilon S$, which concludes the proof of the theorem.

5. Path properties of general stochastic integral processes.

The example following Theorem 4 shows that in the case of Gaussian processes satisfying (*), certain properties of sections of f may be not reflected in the behaviour of the sample paths of these processes. Further, none of the theorems in Section 4 is true for non-random processes. We illustrate this by the following (trivial) example: Let $h:T \to \mathbb{R}$ be *any* function and put $f(t,s)=h(t)[1-2s]$, $s\epsilon(0,1)$. Let Λ be the Lebesgue measure and set $Z(t)=\int_0^1 f(t,s)\Lambda(ds)$. Obviously $Z(t)\equiv 0$ for all t and one can say nothing about the regularities of $f(\cdot,s)$ (i.e. about h). These examples suggest that in order to extend the results of Section 4 to general i.d. processes one should investigate only the pure Poissonian part of an i.d. process (see Maruyama [4] for the decomposition of i.d. processes). The method of symmetrization allows one to remove the deterministic and Gaussian parts of an i.d. process and reduce the problem to the case of a symmetric Poissonian process. We apply this method below, to generalize Proposition 3 of the previous section.

Let $X=\{X(t): t\epsilon T\}$ satisfy (*), where Λ is an arbitrary i.d. random measure with control measure λ. Suppose that for certain linear measurable subspace $C \subset \mathbb{R}^\infty$ and $\{t_n\} \subset T$, with probability one,

(5.1) $$(X(t_1),X(t_2),\ldots) \epsilon C.$$

Let $X' = \{X'(t): t\epsilon T\}$ and $\Lambda'=\{\Lambda'(A): A\epsilon \mathscr{S}\}$ be independent copies of X and Λ, respectively. Put $\tilde{X}(t) = X(t) - X'(t)$ and $\tilde{\Lambda}(A) = \Lambda(A) - \Lambda'(A)$. From (5.1) one gets

(5.2) $$(\tilde{X}(t_1),\tilde{X}(t_2),\ldots) \epsilon C \quad a.s.,$$

and by (*)

$$(5.3) \qquad \{\widetilde{X}(t): t\epsilon T\} \overset{d}{=} \{\int_S f(t,s)\widetilde{\Lambda}(ds): t\epsilon T\}.$$

There exist mutually independent symmetric i.d. random measures

$M = \{M(A): A \in \mathscr{S}\}$ and $W = \{W(A): A \in \mathscr{S}\}$ (defined perhaps on a different

probability space than Λ) such that W is Gaussian, M has no-Gaussian component

and

$$\{\Lambda(A): A\epsilon\mathscr{S}\} \overset{d}{=} \{M(A) + W(A): A \in \mathscr{S}\}.$$

In view of (5.2) and (5.3) we get

$$0 = 2P\{(\widetilde{X}(t_1),\widetilde{X}(t_2),\dots) \notin C\}$$

$$= 2P\{(I_M(t_1) + I_W(t_1), I_M(t_2) + I_W(t_2),\dots) \notin C\}$$

$$\geq P\{(I_M(t_1), I_M(t_2),\dots) \notin C\},$$

by the independence and symmetry, where

$$I_M(t) = \int_S f(t,s)M(ds), \quad I_W(t) = \int_S f(t,s)W(ds), \qquad t\epsilon T.$$

Hence

$$(I_M(t_1),I_M(t_2),\dots) \epsilon C \quad a.s.$$

Applying Proposition 3 (for $X(t) = I_M(t)$) one gets

$$(f(t_1,\cdot), f(t_2,\cdot),\dots) \epsilon C \quad m\text{-a.e.},$$

where m is a control measure of M, and this is the conclusion of Propsoition 3

in the general case. Note that m is absolutely continuous with respect to λ but

not necessarily vice versa.

Proceeding likewise one can generalize Theorems 4 and 6 to arbitrary i.d.

processes by an appropriate replacement of λ by m, a control measure of the

non-Gaussian part of the symmetrization of Λ.

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